DUALITY FOR COLOURED QUANTUM GROUPS [1]

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Abstract

Duality between the coloured quantum group and the coloured quantum algebra corresponding to $GL(2)$ is established. The coloured $L^{\pm}$ functionals are constructed and the dual algebra is derived explicitly. These functionals are then employed to give a coloured generalisation of the differential calculus on quantum $GL(2)$ within the framework of the R-matrix approach.

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1 Introduction

It is well-known [1-3] that the standard quantum group relations can be extended by parametrising the corresponding generators using some continuous 'colour' variables and redefining the associated algebra and coalgebra in a way that all Hopf algebraic properties remain preserved. This results in a 'coloured' extension of a quantum group and the associated R-matrix is a nonadditive type R-matrix.

Non-additive type solutions of the Yang-Baxter equation were first discovered in the study of integrable models by Bazhanov and Stroganov [4]. In the context of quantum groups, a coloured R-matrix solution was first introduced by Murakami [5] and subsequently in [6,7]. In the context of knot theory, Ohtsuki [8] introduced coloured quasitriangular Hopf algebras characterised by the existence of a coloured universal R-matrix. Jordanian deformations also admit coloured extensions [3,9,10] and the associated R-matrix is 'colour' triangular *i.e.*, a coloured extension of the notion of triangularity. More recently, it has been shown [10] that coloured Jordanian deformations can be obtained from the coloured quantum deformations by means of a contraction procedure. Coloured generalisations of quantum groups can also be understood as an application of the twisting procedure, in a manner similar to the multiparameter generalisation of quantum groups.

Even though a considerable interest has been generated in the coloured generalisation of quantum groups, some aspects of their basic algebraic and geometric structure still need thorough investigation. As is the case with ordinary quantum groups, one works on the coloured generalisation of *either* the quantised algebra of functions on a group *or* that of the quantised universal enveloping algebra. It is already well-established [11,12] that the two structures are dual to each other in the uncoloured case. The purpose of this letter is *two-fold*. *Firstly*, we address the problem of duality between coloured quantum groups and coloured quantum algebras. This would then provide us with some more information about the algebraic structure underlying these objects. *Secondly*, it is well-known that in the uncoloured case the problem of duality naturally leads us to formulate a differential

calculus on quantum groups. In the same spirit, we investigate the geometric structure of coloured quantum groups by the construction of a differential calculus. Throughout this letter, we shall focus on the coloured generalisation of the simplest single-parameter quantum deformation of $GL(2)$, denoted as $GL_q^{\lambda,\mu}(2)$. In pursuing our aim, we shall adhere to the convenient R-matrix approach [13,14].

In Section 2, we recall the definition of the coloured quantum group $GL_q^{\lambda,\mu}(2)$. In Section 3, we construct the coloured $L^{\pm}$ functionals for $GL_q^{\lambda,\mu}(2)$ and derive explicitly its dual algebra *i.e.* the coloured quantised universal enveloping algebra and also exhibit its Hopf algebraic structure. Section 4 is devoted to a coloured generalisation of the R-matrix approach (also known as the *constructive* procedure) to construct a differential calculus on the coloured quantum group $GL_q^{\lambda,\mu}(2)$. The Letter concludes with a summary of the results in Section 5.

2 The Coloured Quantum Group $GL_q^{\lambda,\mu}(2)$

For the case of a single-parameter quantum deformation of $GL(2)$ (with deformation parameter q), its 'coloured' version [1] is given by the R-matrix,

$$R_q^{\lambda,\mu} = \begin{pmatrix} q^{1-(\lambda-\mu)} & 0 & 0 & 0 \\ 0 & q^{\lambda+\mu} & 0 & 0 \\ 0 & q-q^{-1} & q^{-(\lambda+\mu)} & 0 \\ 0 & 0 & 0 & q^{1+(\lambda-\mu)} \end{pmatrix} \tag{2.1}$$

which satisfies

$$R_{12}^{\lambda,\mu} R_{13}^{\lambda,\nu} R_{23}^{\mu,\nu} = R_{23}^{\mu,\nu} R_{13}^{\lambda,\nu} R_{12}^{\lambda,\mu} \tag{2.2}$$

the so-called 'coloured' quantum Yang-Baxter equation (CQYBE). The coloured R-matrix provides a nonadditive-type solution $R^{\lambda,\mu} \neq R(\lambda-\mu)$ of the Yang-Baxter equation, which is in general multicomponent and the parameters λ, μ, ν are considered as 'colour' parameters. This gives rise to the coloured RTT relations

$$R_q^{\lambda,\mu} T_{1\lambda} T_{2\mu} = T_{2\mu} T_{1\lambda} R_q^{\lambda,\mu} \tag{2.3}$$

(where $T_{1\lambda} = T_\lambda \otimes \mathbf{1}$ and $T_{2\mu} = \mathbf{1} \otimes T_\mu$) in which the entries of the T matrices carry colour dependence *i.e.* $T_\lambda = \left(\begin{smallmatrix} a_\lambda & b_\lambda \\ c_\lambda & d_\lambda \end{smallmatrix}\right)$, $T_\mu = \left(\begin{smallmatrix} a_\mu & b_\mu \\ c_\mu & d_\mu \end{smallmatrix}\right)$. The coproduct and counit for the coalgebra structure are given by $\Delta(T_\lambda) = T_\lambda \dot{\otimes} T_\lambda$, $\varepsilon(T_\lambda) = \mathbf{1}$ and depend on one colour parameter at a time. By contrast, the algebra structure is more complicated with generators of two different colours appearing simultaneously in the algebraic relations. The full Hopf algebraic structure can be constructed resulting in a coloured extension of the quantum group. Since λ and μ are continuous variables, this implies the coloured quantum group has an infinite number of generators. The quantum determinant $D_\lambda = a_\lambda d_\lambda - r^{-(1+2\lambda)} c_\lambda b_\lambda$ is group-like but not central, even in the case of the single-parameter coloured deformation. This is a crucial difference with the ordinary uncoloured quantum groups. The antipode is given as

$$S(T_\lambda) = D_\lambda^{-1} \begin{pmatrix} d_\lambda & -r^{1+2\lambda} b_\lambda \\ -r^{-1-2\lambda} c_\lambda & a_\lambda \end{pmatrix} \tag{2.4}$$

This results in a coloured generalisation of a quantum group in the framework of the FRT formalism. It should be noted that while the colourless limit $\lambda = \mu = 0$ gives back the ordinary single-parameter deformed quantum group, the monochromatic limit $\lambda = \mu \neq 0$ gives rise to the uncoloured two-parameter deformed quantum group. Analogous to the quantum group case, one can also define the coloured braid group representation as

$$\hat{\mathcal{R}}^{\lambda,\mu} = P\mathcal{R}^{\lambda,\mu} \tag{2.5}$$

where P is the usual permutation matrix. This coloured braid group representation turns out to be a solution of the coloured braided Yang-Baxter equation

$$\hat{R}_{23}^{\lambda,\mu} \hat{R}_{12}^{\lambda,\nu} \hat{R}_{23}^{\mu,\nu} = \hat{R}_{12}^{\mu,\nu} \hat{R}_{23}^{\lambda,\nu} \hat{R}_{12}^{\lambda,\mu} \tag{2.6}$$

3 The Dual Algebra for $GL_q^{\lambda,\mu}(2)$

Here we derive explicitly the algebra dual to the coloured quantum group $GL_q^{\lambda,\mu}(2)$. If we denote generators of the yet unknown dual algebra by $\{A_\lambda, B_\lambda, C_\lambda, D_\lambda\}$ and $\{A_\mu, B_\mu, C_\mu, D_\mu\}$,

then the following pairings hold

$$\langle A_\lambda, T_\lambda\rangle = \langle A_\lambda, T_\mu\rangle = \langle A_\mu, T_\lambda\rangle = \langle A_\mu, T_\mu\rangle = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \tag{3.1}$$

$$\langle B_\lambda, T_\lambda\rangle = \langle B_\lambda, T_\mu\rangle = \langle B_\mu, T_\lambda\rangle = \langle B_\mu, T_\mu\rangle = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \tag{3.2}$$

$$\langle C_\lambda, T_\lambda\rangle = \langle C_\lambda, T_\mu\rangle = \langle C_\mu, T_\lambda\rangle = \langle C_\mu, T_\mu\rangle = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \tag{3.3}$$

$$\langle D_\lambda, T_\lambda\rangle = \langle D_\lambda, T_\mu\rangle = \langle D_\mu, T_\lambda\rangle = \langle D_\mu, T_\mu\rangle = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \tag{3.4}$$

where T_λ and T_μ are the matrices of generators of $GL_q^{\lambda,\mu}(2)$.

3.1 Coloured $L^\pm$ functionals

Recall [11,14] that the $L^\pm$ functionals for the uncoloured single-parameter quantum group $GL_q(2)$ are expressed in terms of the matrices

$$L^+ = c^+ q^{1/2} \begin{pmatrix} q^{H/2} & q^{-1/2}(q - q^{-1})X^+ \\ 0 & q^{-H/2} \end{pmatrix} \tag{3.5}$$

$$L^- = c^- q^{-1/2} \begin{pmatrix} q^{-H/2} & 0 \\ q^{1/2}(q^{-1} - q)X^- & q^{H/2} \end{pmatrix} \tag{3.6}$$

where c^+ and c^- are free parameters and $H = A - D$, $X^+ = C$, $X^- = B$ are the generators of the algebra dual to $GL_q(2)$. The R-matrix for the coloured quantum group $GL_q^{\lambda,\mu}(2)$ can be written as

$$R_{12} = q^{1/2} \begin{pmatrix} q^{-1/2}q^{1-\lambda+\mu} & 0 & 0 & 0 \\ 0 & q^{-1/2}q^{\lambda+\mu} & 0 & 0 \\ 0 & q^{-1/2}(q - q^{-1}) & q^{-1/2}q^{-(\lambda+\mu)} & 0 \\ 0 & 0 & 0 & q^{-1/2}q^{1+\lambda-\mu} \end{pmatrix} \tag{3.7}$$

The corresponding R^+ and R^- matrices read

$$\begin{aligned} R^+ &= c^+ R_{21} \\ &= c^+ q^{1/2} \begin{pmatrix} q^{-1/2} q^{1-\lambda+\mu} & 0 & 0 & 0 \\ 0 & q^{-1/2} q^{-(\lambda+\mu)} & q^{-1/2}(q-q^{-1}) & 0 \\ 0 & 0 & q^{-1/2} q^{\lambda+\mu} & 0 \\ 0 & 0 & 0 & q^{-1/2} q^{1+\lambda-\mu} \end{pmatrix} \end{aligned} \tag{3.8}$$

$$\begin{aligned} R^- &= c^- R_{12}^{-1} \\ &= c^- q^{-1/2} \begin{pmatrix} q^{1/2} q^{-(1-\lambda+\mu)} & 0 & 0 & 0 \\ 0 & q^{1/2} q^{-(\lambda+\mu)} & 0 & 0 \\ 0 & -q^{1/2}(q-q^{-1}) & q^{1/2} q^{\lambda+\mu} & 0 \\ 0 & 0 & 0 & q^{1/2} q^{-(1+\lambda-\mu)} \end{pmatrix} \end{aligned} \tag{3.9}$$

The coloured $L^{\pm}$ functionals can be expressed as

$$L_\lambda^+ = c^+ q^{1/2} \begin{pmatrix} q^{H_\lambda/2} q^{\mu H_\lambda - \lambda H'_\lambda} & q^{-1/2}(q-q^{-1}) C_\lambda \\ 0 & q^{-H_\lambda/2} q^{\mu H_\lambda + \lambda H'_\lambda} \end{pmatrix} \tag{3.10}$$

$$L_\mu^+ = c^+ q^{1/2} \begin{pmatrix} q^{H_\mu/2} q^{\mu H_\mu - \lambda H'_\mu} & q^{-1/2}(q-q^{-1}) C_\mu \\ 0 & q^{-H_\mu/2} q^{\mu H_\mu + \lambda H'_\mu} \end{pmatrix} \tag{3.11}$$

$$L_\lambda^- = c^- q^{-1/2} \begin{pmatrix} q^{-H_\lambda/2} q^{\lambda H_\lambda - \mu H'_\lambda} & 0 \\ q^{1/2}(q^{-1}-q) B_\lambda & q^{H_\lambda/2} q^{\lambda H_\lambda + \mu H'_\lambda} \end{pmatrix} \tag{3.12}$$

$$L_\mu^- = c^- q^{-1/2} \begin{pmatrix} q^{-H_\mu/2} q^{\lambda H_\mu - \mu H'_\mu} & 0 \\ q^{1/2}(q^{-1}-q) B_\mu & q^{H_\mu/2} q^{\lambda H_\mu + \mu H'_\mu} \end{pmatrix} \tag{3.13}$$

where $H_\lambda = A_\lambda - D_\lambda,\ H'_\lambda = A_\lambda + D_\lambda$ and $H_\mu = A_\mu - D_\mu,\ H'_\mu = A_\mu + D_\mu$. Note that each one of $L_\lambda^{\pm}$ and $L_\mu^{\pm}$ depends on *both* the colour parameters λ and μ. The notation $L_\lambda^{\pm}$ implies that the generators of the dual carry λ dependence, and similarly $L_\mu^{\pm}$ implies that the generators of the dual carry μ dependence. The duality pairings are then given by the

action of the functionals $L^{\pm}_{\lambda}$ and $L^{\pm}_{\mu}$ on the T-matrices T_λ and T_μ of the coloured quantum group $GL_q^{\lambda,\mu}(2)$

$$(L^{+}_{\lambda|\mu})^a_b (T_{\lambda|\mu})^c_d = (R^{+})^{ac}_{bd} \tag{3.14}$$

$$(L^{-}_{\lambda|\mu})^a_b (T_{\lambda|\mu})^c_d = (R^{-})^{ac}_{bd} \tag{3.15}$$

The notation $\lambda|\mu$ in the subscript in the above relations means *either* λ *or* μ. So, $T_{\lambda|\mu}$ implies T_λ *or* T_μ and $L^{\pm}_{\lambda|\mu}$ implies $L^{\pm}_{\lambda}$ *or* $L^{\pm}_{\mu}$. If we define $X^{+}_{\lambda} = C_\lambda$, $X^{+}_{\mu} = C_\mu$, $X^{-}_{\lambda} = B_\lambda$, $X^{-}_{\mu} = B_\mu$, then the spin-$\frac{1}{2}$ representation ρ can be given as

$$\rho(H_\lambda) = \rho(H_\mu) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \tag{3.16}$$

$$\rho(X^{+}_{\lambda}) = \rho(X^{+}_{\mu}) = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \tag{3.17}$$

$$\rho(X^{-}_{\lambda}) = \rho(X^{-}_{\mu}) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \tag{3.18}$$

$$\rho(H'_\lambda) = \rho(H'_\mu) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \tag{3.19}$$

and

$$\rho((X^{\pm}_{\lambda|\mu})^2) = 0 \qquad\qquad \rho(H^2_{\lambda|\mu}) = 1 = \rho({H'}^2_{\lambda|\mu}) \tag{3.20}$$

For vanishing colour parameters, the coloured $L^{\pm}$ functionals reduce to the ordinary $L^{\pm}$ functionals for $GL_q(2)$.

3.2 Coloured RLL relations

Similar to the uncoloured case, the commutation relations of the algebra dual to a coloured quantum group can be obtained using the modified or the *coloured RLL* relations using the coloured $L^{\pm}$ functionals of the previous section. Bearing close resemblence to the coloured RTT relations, these can be defined as

$$R_{12} L^{\pm}_{2\lambda} L^{\pm}_{1\mu} = L^{\pm}_{1\mu} L^{\pm}_{2\lambda} R_{12} \tag{3.21}$$

$$R_{12} L^{+}_{2\lambda} L^{-}_{1\mu} = L^{-}_{1\mu} L^{+}_{2\lambda} R_{12} \tag{3.22}$$

where $L^{\pm}_{1\mu} = L^{\pm}_{\mu} \otimes \mathbf{1}$ and $L^{\pm}_{2\lambda} = \mathbf{1} \otimes L^{\pm}_{\lambda}$. Using the above formulae, we obtain the commutation relations between the generating elements of the algebra dual to the coloured quantum group $GL^{\lambda,\mu}_q(2)$.

$$\begin{array}{lll} [A_\lambda, B_\mu] = B_\mu & [D_\lambda, B_\mu] = -B_\mu & \\ [A_\lambda, C_\mu] = -C_\mu & [D_\lambda, C_\mu] = C_\mu & \\ [A_\lambda, D_\mu] = 0 & [H_\lambda, H_\mu] = 0 & [H'_\lambda, \bullet] = 0 \end{array} \tag{3.23}$$

$$q^{-(\lambda+\mu)} C_\lambda B_\mu - q^{\lambda+\mu} B_\mu C_\lambda = \frac{q^{\lambda H_\mu + \mu H_\lambda}}{q - q^{-1}} \left[q^{-\frac{1}{2}(H_\lambda + H_\mu)} q^{\lambda H'_\lambda - \mu H'_\mu} - q^{\frac{1}{2}(H_\lambda + H_\mu)} q^{-\lambda H'_\lambda + \mu H'_\mu} \right] \tag{3.24}$$

$$\begin{array}{rcl} A_\lambda A_\mu & = & A_\mu A_\lambda \\ B_\lambda B_\mu & = & q^{2(\mu-\lambda)} B_\mu B_\lambda \\ C_\lambda C_\mu & = & q^{2(\lambda-\mu)} C_\mu C_\lambda \\ D_\lambda D_\mu & = & D_\mu D_\lambda \end{array} \tag{3.25}$$

where H_λ and H'_λ are as before. The relations satisfy the $\lambda \leftrightarrow \mu$ exchange symmetry. The associated coproduct of the elements of the dual algebra is given by

$$\Delta(A_\lambda) = A_\lambda \otimes \mathbf{1} + \mathbf{1} \otimes A_\lambda \tag{3.26}$$

$$\Delta(B_\lambda) = B_\lambda \otimes q^{A_\lambda - D_\lambda} + \mathbf{1} \otimes B_\lambda \tag{3.27}$$

$$\Delta(C_\lambda) = C_\lambda \otimes q^{A_\lambda - D_\lambda} + \mathbf{1} \otimes C_\lambda \tag{3.28}$$

$$\Delta(D_\lambda) = D_\lambda \otimes \mathbf{1} + \mathbf{1} \otimes D_\lambda \tag{3.29}$$

The counit $\varepsilon(Y_\lambda) = 0$; where $Y_\lambda = \{A_\lambda, B_\lambda, C_\lambda, D_\lambda\}$ and the antipode is given as

$$S(A_\lambda) = -A_\lambda \tag{3.30}$$

$$S(B_\lambda) = -B_\lambda q^{-(A_\lambda - D_\lambda)} \tag{3.31}$$

$$S(C_\lambda) = -C_\lambda q^{-(A_\lambda - D_\lambda)} \tag{3.32}$$

$$S(D_\lambda) = -D_\lambda \tag{3.33}$$

The Hopf algebra structure is also invariant under the $\lambda \leftrightarrow \mu$ exchange symmetry. Thus we have defined a new single-parameter coloured quantum algebra corresponding to $gl(2)$, which in the monochromatic limit defines the standard uncoloured two-parameter quantum algebra for $gl(2)$. Note that while the Hopf algebra structure underlying a coloured quantum group and its dual is infinite dimensional, it can be couched in a more familiar form of the finite dimensional case as shown above. This makes it plausible to investigate the topological aspect of duality for coloured quantum groups along the lines of the treatment furnished in [15] for infinite dimensional Hopf algebras.

4 Differential Calculus on $GL_q^{\lambda,\mu}(2)$

We now proceed to the construction of a differential calculus on the coloured quantum group $GL_q^{\lambda,\mu}(2)$. Our preferred approach to this problem is the R-matrix formalism, which we shall now generalise to the case of coloured quantum groups.

4.1 One-forms

Analogous to the standard uncoloured quantum group, a bimodule Γ (space of quantum one-forms ω) is characterised by the commutation relations between ω and $a_{\lambda(\mu)} \in \mathcal{A}$, the *coloured* quantum group or Hopf algebra under consideration,

$$\omega a_{\lambda(\mu)} = (\mathbf{1} \otimes f_{\lambda,\mu})\Delta(a_{\lambda(\mu)})\omega \tag{4.1}$$

where $a_{\lambda(\mu)}$ denotes a_λ (*respectively* a_μ) and the linear functional $f_{\lambda,\mu}$ is now defined in terms of the coloured $L^\pm$ matrices as

$$f_{\lambda,\mu} = S(L^+_{\lambda|\mu})L^-_{\lambda|\mu} \tag{4.2}$$

For the coloured quantum group $GL_q^{\lambda,\mu}(2)$, these read

$$S(L^+_{\lambda(\mu)}) = (c^+ q^{1/2})^{-1} \begin{pmatrix} q^{-H_{\lambda(\mu)}/2} q^{-\mu H_{\lambda(\mu)} + \lambda H'_{\lambda(\mu)}} & S(l^+_{\lambda(\mu)})_{12} \\ 0 & q^{H_{\lambda(\mu)}/2} q^{-\mu H_{\lambda(\mu)} - \lambda H'_{\lambda(\mu)}} \end{pmatrix} \tag{4.3}$$

with

$$S(l^+_{\lambda(\mu)})_{12} = -q^{-1/2}(q-q^{-1})q^{-H_{\lambda(\mu)}/2}q^{-\mu H_{\lambda(\mu)}+\lambda H'_{\lambda(\mu)}}C_{\lambda(\mu)}q^{H_{\lambda(\mu)}/2}q^{-\mu H_{\lambda(\mu)}-\lambda H'_{\lambda(\mu)}}$$

$$L^-_{\lambda(\mu)} = c^- q^{-1/2}\begin{pmatrix} q^{-H_{\lambda(\mu)}/2}q^{\lambda H_{\lambda(\mu)}-\mu H'_{\lambda(\mu)}} & 0 \\ q^{1/2}(q^{-1}-q)B_{\lambda(\mu)} & q^{H_{\lambda(\mu)}/2}q^{\lambda H_{\lambda(\mu)}+\mu H'_{\lambda(\mu)}} \end{pmatrix} \tag{4.4}$$

where $H_{\lambda(\mu)} = A_{\lambda(\mu)} - D_{\lambda(\mu)}$ and $H'_{\lambda(\mu)} = A_{\lambda(\mu)} + D_{\lambda(\mu)}$ Thus, we have

$$\omega a_{\lambda(\mu)} = [(\mathbf{1}\otimes S(L^+_{\lambda|\mu})L^-_{\lambda|\mu})\Delta(a_{\lambda(\mu)})]\omega \tag{4.5}$$

In terms of components, this can be written as

$$\omega_{ij}a_{\lambda(\mu)} = [(\mathbf{1}\otimes S(l^+_{(\lambda|\mu)ki})l^-_{(\lambda|\mu)jl})\Delta(a_{\lambda(\mu)})]\omega_{kl} \tag{4.6}$$

using the expressions $L^\pm = l^\pm_{ij}$ and $\omega = \omega_{ij}$ where $i,j = 1,2$. Using these relations, we obtain the commutation relations of all the left-invariant one-forms with the elements of the coloured quantum group $GL_q^{\lambda,\mu}(2)$ as follows

$$\begin{aligned}
\omega^1 a_{\lambda(\mu)} &= \mathsf{s}q^{-2+2(\lambda-\mu)}a_{\lambda(\mu)}\omega^1 && (4.7)\\
\omega^+ a_{\lambda(\mu)} &= \mathsf{s}q^{-1+2\lambda}a_{\lambda(\mu)}\omega^+ && (4.8)\\
\omega^- a_{\lambda(\mu)} &= \mathsf{s}q^{-1-2\mu}a_{\lambda(\mu)}\omega^- + \mathsf{s}(q^{-2}-1)q^{(\lambda-\mu)}b_{\lambda(\mu)}\omega^1 && (4.9)\\
\omega^2 a_{\lambda(\mu)} &= \mathsf{s}a_{\lambda(\mu)}\omega^2 + \mathsf{s}(q^{-1}-q)q^{(\lambda+\mu)}b_{\lambda(\mu)}\omega^+ && (4.10)
\end{aligned}$$

$$\begin{aligned}
\omega^1 b_{\lambda(\mu)} &= \mathsf{s}b_{\lambda(\mu)}\omega^1 && (4.11)\\
\omega^+ b_{\lambda(\mu)} &= \mathsf{s}q^{-1+2\mu}b_{\lambda(\mu)}\omega^+ + \mathsf{s}(q^{-2}-1)q^{(\lambda-\mu)}a_{\lambda(\mu)}\omega^1 && (4.12)\\
\omega^- b_{\lambda(\mu)} &= \mathsf{s}q^{-1-2\lambda}b_{\lambda(\mu)}\omega^- && (4.13)\\
\omega^2 b_{\lambda(\mu)} &= \mathsf{s}q^{-2+2(\mu-\lambda)}b_{\lambda(\mu)}\omega^2 + \mathsf{s}(q^{-1}-q)q^{-(\lambda+\mu)}a_{\lambda(\mu)}\omega^- && (4.14)\\
&\quad +\mathsf{s}(q^{-1}-q)^2 b_{\lambda(\mu)}\omega^1
\end{aligned}$$

$$\omega^1 c_{\lambda(\mu)} = \mathsf{s} q^{-2+2(\lambda-\mu)} c_{\lambda(\mu)} \omega^1 \tag{4.15}$$

$$\omega^+ c_{\lambda(\mu)} = \mathsf{s} q^{-1+2\lambda} c_{\lambda(\mu)} \omega^+ \tag{4.16}$$

$$\omega^- c_{\lambda(\mu)} = \mathsf{s} q^{-1-2\mu} c_{\lambda(\mu)} \omega^- + \mathsf{s}(q^{-2}-1) q^{(\lambda-\mu)} d_{\lambda(\mu)} \omega^1 \tag{4.17}$$

$$\omega^2 c_{\lambda(\mu)} = \mathsf{s} c_{\lambda(\mu)} \omega^2 + \mathsf{s}(q^{-1}-q) q^{(\lambda+\mu)} d_{\lambda(\mu)} \omega^+ \tag{4.18}$$

$$\omega^1 d_{\lambda(\mu)} = \mathsf{s} d_{\lambda(\mu)} \omega^1 \tag{4.19}$$

$$\omega^+ d_{\lambda(\mu)} = \mathsf{s} q^{-1+2\mu} d_{\lambda(\mu)} \omega^+ + \mathsf{s}(q^{-2}-1) q^{(\lambda-\mu)} c_{\lambda(\mu)} \omega^1 \tag{4.20}$$

$$\omega^- d_{\lambda(\mu)} = \mathsf{s} q^{-1-2\lambda} d_{\lambda(\mu)} \omega^- \tag{4.21}$$

$$\begin{aligned} \omega^2 d_{\lambda(\mu)} &= \mathsf{s} q^{-2+2(\mu-\lambda)} d_{\lambda(\mu)} \omega^2 + \mathsf{s}(q^{-1}-q) q^{-(\lambda+\mu)} c_{\lambda(\mu)} \omega^- \\ &\quad + \mathsf{s}(q^{-1}-q)^2 d_{\lambda(\mu)} \omega^1 \end{aligned} \tag{4.22}$$

where $\omega^1 = \omega_{11}$, $\omega^+ = \omega_{12}$, $\omega^- = \omega_{21}$, $\omega^2 = \omega_{22}$ and $\mathsf{s} = (c^+)^{-1} c^-$.

4.2 Vector fields

The left-invariant vector fields χ_{ij} on $\mathcal{A}$ are given by the expression

$$\chi_{ij} = S(l^+_{(\lambda|\mu)ik}) l^-_{(\lambda|\mu)kj} - \delta_{ij} \varepsilon \tag{4.23}$$

or simply

$$\chi = S(L^+_{\lambda|\mu}) L^-_{\lambda|\mu} - \mathbf{1} \varepsilon \tag{4.24}$$

On elements of $GL_q^{\lambda,\mu}(2)$, the vector fields act as

$$\chi_{ij} a_{\lambda(\mu)} = (S(l^+_{(\lambda|\mu)ik}) l^-_{(\lambda|\mu)kj} - \delta_{ij} \varepsilon) a_{\lambda(\mu)} \tag{4.25}$$

$$\chi_{ij} a_{\lambda(\mu)} = \langle S(l^+_{(\lambda|\mu)ik}) l^-_{(\lambda|\mu)kj}, a_{\lambda(\mu)} \rangle - \delta_{ij} \varepsilon(a_{\lambda(\mu)}) \tag{4.26}$$

for $a_{\lambda(\mu)} \in GL_q^{\lambda,\mu}(2)$. We obtain explicitly the following

$$\begin{aligned}
&\chi_1(a_{\lambda|\mu}) = \mathsf{s}q^{-2+2(\lambda-\mu)} - 1 && \chi_1(b_{\lambda|\mu}) = 0 \\
&\chi_+(a_{\lambda|\mu}) = 0 && \chi_+(b_{\lambda|\mu}) = 0 \\
&\chi_-(a_{\lambda|\mu}) = 0 && \chi_-(b_{\lambda|\mu}) = \mathsf{s}(q^{-1} - q)q^{-(\lambda+\mu)} \\
&\chi_2(a_{\lambda|\mu}) = \mathsf{s} - 1 && \chi_2(b_{\lambda|\mu}) = 0
\end{aligned} \tag{4.27}$$

$$\begin{aligned}
&\chi_1(c_{\lambda|\mu}) = 0 && \chi_1(d_{\lambda|\mu}) = \mathsf{s}(q^{-1} - q)^2 + \mathsf{s} - 1 \\
&\chi_+(c_{\lambda|\mu}) = \mathsf{s}(q^{-1} - q)q^{-(\lambda+\mu)} && \chi_+(d_{\lambda|\mu}) = 0 \\
&\chi_-(c_{\lambda|\mu}) = 0 && \chi_-(d_{\lambda|\mu}) = 0 \\
&\chi_2(c_{\lambda|\mu}) = 0 && \chi_2(d_{\lambda|\mu}) = \mathsf{s}q^{-2+2(\mu-\lambda)} - 1
\end{aligned} \tag{4.28}$$

where $\chi_0 = \chi_{11}$, $\chi_+ = \chi_{12}$, $\chi_- = \chi_{21}$, $\chi_2 = \chi_{22}$, and $\mathsf{s} = (c^+)^{-1}c^-$. The left convolution products are given as

$$\begin{aligned}
&\chi_1 * a_{\lambda(\mu)} = (\mathsf{s}q^{-2+2(\lambda-\mu)} - 1)a_{\lambda(\mu)} && \chi_1 * b_{\lambda(\mu)} = (\mathsf{s}(q^{-1} - q)^2 + \mathsf{s} - 1)b_{\lambda(\mu)} \\
&\chi_+ * a_{\lambda(\mu)} = (\mathsf{s}(q^{-1} - q)q^{(\lambda+\mu)})b_{\lambda(\mu)} && \chi_+ * b_{\lambda(\mu)} = 0 \\
&\chi_- * a_{\lambda(\mu)} = 0 && \chi_- * b_{\lambda(\mu)} = (\mathsf{s}(q^{-1} - q)q^{-(\lambda+\mu)})a_{\lambda(\mu)} \\
&\chi_2 * a_{\lambda(\mu)} = (\mathsf{s} - 1)a_{\lambda(\mu)} && \chi_2 * b_{\lambda(\mu)} = (\mathsf{s}q^{-2+2(\mu-\lambda)} - 1)b_{\lambda(\mu)}
\end{aligned} \tag{4.29}$$

$$\begin{aligned}
&\chi_1 * c_{\lambda(\mu)} = (\mathsf{s}q^{-2+2(\lambda-\mu)} - 1)c_{\lambda(\mu)} && \chi_1 * d_{\lambda(\mu)} = (\mathsf{s}(q^{-1} - q)^2 + \mathsf{s} - 1)d_{\lambda(\mu)} \\
&\chi_+ * c_{\lambda(\mu)} = (\mathsf{s}(q^{-1} - q)q^{(\lambda+\mu)})d_{\lambda(\mu)} && \chi_+ * d_{\lambda(\mu)} = 0 \\
&\chi_- * c_{\lambda(\mu)} = 0 && \chi_- * d_{\lambda(\mu)} = (\mathsf{s}(q^{-1} - q)q^{-(\lambda+\mu)})c_{\lambda(\mu)} \\
&\chi_2 * c_{\lambda(\mu)} = (\mathsf{s} - 1)c_{\lambda(\mu)} && \chi_2 * d_{\lambda(\mu)} = (\mathsf{s}q^{-2+2(\mu-\lambda)} - 1)d_{\lambda(\mu)}
\end{aligned} \tag{4.30}$$

4.3 Exterior derivatives

Using the formula $\mathbf{d}a_{\lambda(\mu)} = \sum_i(\chi_i * a_{\lambda(\mu)})\omega^i$ for $a_{\lambda(\mu)} \in \mathcal{A}$, we obtain the action of the exterior derivative on the generating elements of $GL_q^{\lambda,\mu}(2)$

$$\mathbf{d}a_{\lambda(\mu)} = (\mathsf{s}q^{-2+2(\lambda-\mu)} - 1)a_{\lambda(\mu)}\omega^1 + \mathsf{s}(q^{-1} - q)q^{\lambda+\mu}b_{\lambda(\mu)}\omega^+ + (\mathsf{s} - 1)a_{\lambda(\mu)}\omega^2 \tag{4.31}$$

$$\mathbf{d}b_{\lambda(\mu)} = (\mathsf{s}(q^{-1} - q)^2 + \mathsf{s} - 1)b_{\lambda(\mu)}\omega^1 + \mathsf{s}(q^{-1} - q)q^{-(\lambda+\mu)}a_{\lambda(\mu)}\omega^- + (\mathsf{s}q^{-2+2(\mu-\lambda)} - 1)b_{\lambda(\mu)}\omega^2 \tag{4.32}$$

$$\mathbf{d}c_{\lambda(\mu)} = (\mathsf{s}q^{-2+2(\lambda-\mu)} - 1)c_{\lambda(\mu)}\omega^1 + \mathsf{s}(q^{-1} - q)q^{\lambda+\mu}d_{\lambda(\mu)}\omega^+ + (\mathsf{s} - 1)c_{\lambda(\mu)}\omega^2 \tag{4.33}$$

$$\mathbf{d}d_{\lambda(\mu)} = (\mathsf{s}(q^{-1} - q)^2 + \mathsf{s} - 1)d_{\lambda(\mu)}\omega^1 + \mathsf{s}(q^{-1} - q)q^{-(\lambda+\mu)}c_{\lambda(\mu)}\omega^- + (\mathsf{s}q^{-2+2(\mu-\lambda)} - 1)d_{\lambda(\mu)}\omega^2 \tag{4.34}$$

$\mathbf{d}\mathcal{A}$ generates Γ as a left $\mathcal{A}$-module. This then defines a first order differential calclulus $(\Gamma, \mathbf{d})$ on $GL_q^{\lambda,\mu}(2)$. Note that because the colour parameters λ and μ are continuously varying, the differential calculus obtained is infinite dimensional. It can be checked that the differential calculus on the uncoloured single-parameter quantum group $GL_q(2)$ is recovered in the colourless limit $\lambda = \mu = 0$. Furthermore, in the monochromatic limit $\lambda = \mu \neq 0$, the differential calculus reduces to that of the uncoloured two-parameter quantum group $GL_{p,q}(2)$.

5 Conclusions

In this work, we have given a description of the duality between a coloured quantum group and a coloured quantum algebra employing the R-matrix approach. In particular, we have derived explicitly the algebra dual to the coloured quantum group $GL_q^{\lambda,\mu}(2)$. Following the duality, we have also constructed a differential calculus on $GL_q^{\lambda,\mu}(2)$ by providing a coloured generalisation of the R-matrix (or the *constructive*) procedure. Both, the dual algebra as well as the differential calculus for $GL_q^{\lambda,\mu}(2)$ reduce to that of the ordinary $GL_q(2)$ in the colourless limit and to the uncoloured two-parameter quantum group in the monochromatic

limit. The results obtained are general enough to be applicable to multi-parametric and higher dimensional coloured quantum groups.

It would be interesting to give dual basis for a coloured quantum group and to investigate differential calculi on its quantum planes.

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